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Nonlinear sigma model and Yang-Mills theory through non-critical string

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Abstract

We consider two-dimensional nonlinear sigma model from the viewpoint of the holography, which has been applied to the study of the Yang-Mills theory, based on the non-critical string theory. We can see the renormalization group flows for both the nonlinear sigma model and the Yang-Mills theory at the same time, and the two theories are intimately related through a kind of dual relation of the coupling constants. We address the running behaviors of these coupling constants and also the asymptotic freedom of the Yang-Mills theory.

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1 Introduction

Following the conjecture of the duality [1, 2, 3] between the super Yang-Mills theory and super-string theory, many people approached to the Yang-Mills theory in the context of the string theory. And an approach to the non-supersymmetric Yang-Mills theory has been proposed in [4] by using the non-critical string theory based on the supersymetric Liouville theory. In this theory, the space-time fermions are removed by taking the GSO projection as in the type 0 string theory which has no space-time supersymmetry [5, 6]. Instead, two kinds of RR-fields and tachyon-field appear in the bulk space other than the usual bosons of the type II string theory. While the tachyon in the world volume of the D-brane can be removed by imposing the GSO projection on the open string sector. Then the Yang-Mills field and some scalar fields are retained on the D-brane world volume. This formulation could be extendable to any dimension smaller than ten, and the five-dimensional case has been considered previously to study the pure 4d Yang-Mills theory [7, 8, 9, 10].

On the other hand, it could be possible to identify the world-sheet action of the non-critical string with the fully quantized 2d sigma model coupled to the quantum gravity [11]. The critical behaviors of the sigma model are obtained from the classical solutions of the low-energy effective action of the non-critical string by noticing the relation of the Liouville field and the mass-scale of the 2d sigma model. If this approach to the 2d sigma model is successful, then it would be possible to determine the critical behaviors of two different theories, the Yang-Mills theory and 2d sigma model, simultaneously by one classical solution of the non-critical string theory with D-brane background. The Yang-Mills fields live in the bulk space where D-branes are accumulated, and the sigma model is defined on the world-sheet of the non-critical string. These two theories, which live on different dimensional and independent world, are connected by a higher dimensional gravitational theory or a closed string theory. [2] The 2d sigma model has been examined by a motivation that its properties are similar to the 4d Yang-Mills theory, i.e. asymptotic freedom and instantons e.t.c.. From the viewpoint of the non-critical string theory, they are related more intimately. They could be studied as a dynamically coupled system through the classical equations, which represent the renormalization group equations of the coupled system, of the non-critical string theory.

Previously, the bosonic case of the non-critical string has been considered under the linear dilaton background [11], but the discussions were formal since the central charge is larger than one for the sigma model. The problem to be resolved is the possible instability of the tachyon in the bulk. This problem could be however evaded by considering an appropriate potential [14, 13] or a curved space-time [15]. When we consider the non-critical string based on the supersymmetric world-sheet action, the RR-fields should be added to the condensed background fields, and the possibility to resolve the tachyon instability is extended [16]. At least, this problem is resolved by considering the AdS bulk space generated by the condensation of RR-fields. More

[2] An approach to this problem has recently been proposed in [12] from the viewpoint of the confromal field theory by exploiting linear renormalization group equations near the conformal limit.

important thing provided by the RR-fields is the existence of the Yang-Mills theory on the boundary of this curved space. For these two reasons, it would be meaningful to consider the non-critical string based on the supersymmetric world-sheet action.

Our purpose is to address the critical behaviours of both theories, the Yang-Mills theory and 2d sigma model, through the non-critical string theory constructed by the supersymmetric world-sheet action. The mass scale of the running coupling constants for both theories is given by the Liouville field. The solutions of the low-energy effective action are obtained as functions of Liouville field in order to obtain the renormalization group equations of the two theories.

In the next section, we briefly review the relation of the 2d sigma model coupled to quantum gravity and the non-critical string theory. And the role of the Liouville field as a mass scale is also reviewed. The low-energy effective action and the gravitational equations to be solved are given in the section three. In section four, the critical behaviours of the sigma model are addressed in the linear dilaton vacuum, where D-branes are absent, according to the equations given in the previous section. When D-branes are added, the Yang-Mills theory appears and couples to the sigma model. In section five, we consider the fixed points in this case and finite coupling constants are given. A kind of dual relation of the coupling constants for the two theories is shown. In obtaining these solutions, the RR-fields are essential. Further, the running behaviours of the coupling constants of both theories are addressed. In section six, we show several solutions with asymptotic freedom for either theory. In these cases, the $d+1$ dimensional space deviates largely from the AdS_{d+1} which appears near the horizon of the D-branes in type IIB string model. In the final section, concluding remarks and discussions are given.

2 2d sigma model coupled to gravity and mass scale

Here, we briefly review a method to obtain the renormalization group equation of the sigma model coupled to quantum gravity in two dimension according to [17, 18, 11].

First, we consider a theory before coupling to quantum gravity and a simple derivation of the β-function in terms of conformal field theory [19]. Let S_0 denote the action of the conformal invariant theory, and let J denote a marginal operator. The total theory will be written by:

$$S = S_0 + f \int d^2 z J, \tag{1}$$

and this system will deviate away from the conformal fixed point given by S_0. Denote the short distance cut-off $\hat{a}$, and assume that J has the operator product expansion, i.e.

$$J(r)J(0) \sim \frac{c}{r^2} J(0), \qquad r \to 0. \tag{2}$$

This implies

$$f^n \int d^2 z_1 d^2 z_2 \left\langle \cdots J(z_1) J(z_2) \cdots \right\rangle_{S_0} \sim -2\pi c \cdot \log \hat{a} f^n \int d^2 z \left\langle \cdots J(z) \cdots \right\rangle_{S_0}. \tag{3}$$

A change in cut-off $\hat{a} \to \hat{a}(1+dl)$ can be compensated by a change $f \to f - dl\pi c f^2$ in the lower order term of the perturbative expansion and this leads to the β-function

$$\beta(f) = -\frac{df}{dl} = \pi c f^2 + O(f^3). \tag{4}$$

Next, let us consider the situation where the theory is coupled to quantum gravity. We work in conformal gauge. The metric is decomposed in a fiducial background metric and the Liouville field ρ by

$$g_{\alpha\beta} = \hat{g}_{\alpha\beta} e^{2\rho}. \tag{5}$$

At the conformal point, where $f = 0$, the coupled theory is given in Ref. [18] as the sum of the Liouville action and S_0 described by $\hat{g}_{\alpha\beta}$. The Liouville part of the theory can be written as [18, 20]

$$S_L = \frac{1}{8\pi}\int d^2z\sqrt{\hat{g}}\left(\hat{g}^{\alpha\beta}\partial_\alpha\rho\partial_\beta\rho - Q\hat{R}\rho + \mu e^{\alpha\rho}\right), \tag{6}$$

where

$$\alpha = \begin{cases} -\frac{1}{2}(Q-\sqrt{Q^2-8}) & ,Q=\sqrt{\frac{25-c}{3}} \quad \text{for Bosonic} \\ -\frac{1}{2}(Q-\sqrt{Q^2-4}) & ,Q=\sqrt{\frac{9-c}{2}} \quad \text{for Supersymmetric.} \end{cases} \tag{7}$$

There are two ultra-violet cut-off: $\hat{a}$ defined in terms of the fiducial metric $\hat{g}_{\alpha\beta}$ and the physical cut-off a defined by

$$ds^2 = e^{\alpha\rho}\hat{g}_{\alpha\beta}dz^\alpha dz^\beta > a^2. \tag{8}$$

The theory must be independent of the cut-off $\hat{a}$ since the fiducial metric is arbitrary: the β-function must vanish. If we consider the theory defined by (1) ($f \neq 0$ case) which has a non-vanishing β-function before coupling to gravity, we have a dependence of $\hat{a}$. Then this dependence should be eliminated by introducing new couplings between the Liouville field and the matter fields such that all cut-off dependence of $\hat{a}$ cancels order by order of f. In this way, the fully quantized action can be obtained for the theory of interacting matter fields coupled to the gravity.

The total action $S_L + S_{\text{matter}}$ can be written by a more general form of non-linear sigma model:

$$S_{eff} = \frac{1}{4\pi}\int d^2z\sqrt{\hat{g}}\left[\frac{1}{2}G_{\mu\nu}(X)\hat{g}^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu + \hat{R}\Phi(X) + T(X)\right], \tag{9}$$

where the Liouville field is represented by one of X^μ; $X^\mu = (X^0, X^i) = (\rho, X^i)$. In the terminology of string theory, $G_{\mu\nu}(X)$, $\Phi(X)$ and $T(X)$ represent the metric, dilaton and tachyon, respectively, and they are determined such that the theory is conformal invariant.

Although the β-function in terms of the unphysical cut-off $\hat{a}$ is zero by construction, we can see a slightly modified form of the original β-function (4) from the change of the action generated by a change of the physical cut-off a defined by (8).

Consider a change $a \to a(1+dl)$ of the physical cut-off. According to (8) this leads to a change $\rho \to \rho + 2dl/\alpha$. The "running" of the coupling constants is seen by absorbing the shift $\rho \to \rho + 2dl/\alpha$ in a redefinition of the coupling constants f [21, 22]. This requirement determines the scale-dependence of f or its β-function. To $O(f^3)$ the modified β-function, $\beta_G(f)$, defined as $-df/dl$, is obtained, and it is related to the β-function, $\beta_(f)$, obtained before coupling to gravity as

$$\beta_G(f) = \frac{2}{\alpha Q}\beta(f). \tag{10}$$

The modification by the pre-factor $\frac{2}{\alpha Q}$ is called as "gravitational dressing" of the β function. This is also seen in [21, 22] for $O(N)$ non-linear sigma model and sine-Gordon model.

A more efficient way to obtain the fully quantized action or the renormalization group equations is to consider in the framework of the non-critical string. The total action S_{eff} given in (9) can be interpreted as the world-sheet action of the non-critical string in the background specified by $G_{\mu\nu}(X)$, $\Phi(X)$ and $T(X)$, which are given as a solution of classical equations of the low-energy effective action of the string theory. Here, the solutions should be obtained as

$$G_{ij}(X) = G_{ij}(\rho, X^i), \quad \Phi(X) = \Phi(\rho), \quad T(X) = T(\rho), \tag{11}$$

and $G_{ij}(0, X^i)$ represents the original part of the sigma model. In other words, the solutions are restricted to the form which could represent the renormalization group equations of the coupling constants of the original field theory coupled to the quantum gravity. The mass-scale of the renormalization group is given here by the Liouville field ρ.

If this procedure is successful to see the critical behaviour of the 2d sigma model, this would be regarded as a kind of holography. Namely, the renormalization group flows of 2d sigma model could be found from higher dimensional gravity. Further analysis in this context is given in section four in the linear dilaton vacuum with non-trivial potential of the tachyon.

We consider this program in a more interesting case, where D-branes are contained in the bulk configurations. In this case, the gravity in the bulk is the dual of the Yang-Mills theory on the branes and e^{Φ} represents the coupling constant of the Yang-Mills theory. Then we can see the running of both coupling constants simultaneously by solving the same gravitational equations under the assumption that ρ represents the mass scale for both field theories. One more merit to consider D-branes is that it would be possible to extend the sigma model to the case of $c > 1$ since the background has a curvature to stabilize the tachyon.

3 Effective action and equations to be solved

Let us consider here the super-symmetric case of the non-linear sigma model which has $d+n=D-1$ boson fields; The boson part of the action is written as

$$S = \frac{1}{8\pi}\int d^2z\sqrt{g}\left\{\sum_{\mu=1}^{d}\partial_\beta\phi^\mu\partial^\beta\phi^\mu + \sum_{i,j=1}^{n}\hat{G}_{ij}(\phi)\partial_\beta\phi^i\partial^\beta\phi^j\right\}, \tag{12}$$

where $\hat{G}_{ij}(\phi)$ denotes the S^n metric, and the fermionic parts are neglected since they are not used here. The full super-symmetric and general form of action would be found in [23]. This model consists of d-free fields and n interacting fields, which form the $O(n+1)$ non-linear sigma model here. The sector of the free fields are prepared for the space-time of the D-branes where Yang-Mills fields are living. The interesting case is $d=4$, but we consider in general d here. In the 2d world-sheet action the parameters of the Yang-Mills theory can not be seen until the theory is coupled to the quantum gravity. When the model is coupled to 2d quantum gravity and quantized in the conformal gauge, we get (ignoring as usual the ghost terms) the following boson part of the quantized action as

$$\begin{aligned} S(\hat{g}) &= \frac{1}{8\pi}\int d^2z\sqrt{\hat{g}}\Big\{G_{00}(\rho)\partial_\beta\rho\partial^\beta\rho + \hat{R}\Phi(\rho) + G_{MN}(\rho,\phi)\partial_\beta\phi^M\partial^\beta\phi^N \\ &\qquad + (RR-\mathrm{bg})\Big\}, \end{aligned} \tag{13}$$

where $M,N=1\sim D-1$, $G_{MN}(\rho,\phi)=\{G_{\mu\nu}(\rho),G_{ij}(\rho,\phi)\}$ and ρ denotes the Liouville field. The last term $(RR-\mathrm{bg})$ represents the part depending of the RR-background fields which appear since we are considering super-symmetric world sheet action of the corresponding string model. $\Phi(\rho)$, $G_{00}(\rho)$, $G_{MN}(\rho,\phi)$ and $RR-\mathrm{bg}$ are determined such that the above action is conformal invariant as mentioned in the previous section. From the viewpoint of the holography, we comment on these functions: (i) When the D-branes are present, $e^{\Phi(\rho)}$ represents the Yang-Mills coupling constant. (ii) As for $G_{MN}(\rho,\phi)$, its sigma model part can be denoted as $G_{ij}(\rho,\phi)=e^{2C(\rho)}\hat{G}_{ij}(\phi)$ according to the principle mentioned in the previous section, and $e^{2C(\rho)}$ represents the running coupling constant of the sigma model. We must notice here that it is controlled by the Yang-Mills theory or D-branes. The free field part is also non-trivial in the presence of D-branes, and it is denoted here as $G_{\mu\nu}(\rho)=e^{2A(\rho)}\eta_{\mu\nu}$. This is the reflection of the fact that the sector of the Yang-Mills theory interacts with the sigma model sector via Liouville fields ρ.

This action represents the boson part of the super-symmetric world-sheet action of the non-critical string which couples to the background defined by $G_{MN}(\rho,\phi)$, $\Phi(\rho)$ and $RR-bg$. Here we suppose a string theory of type 0 in which there is no space-time super-symmetry but modular invariance of the loop amplitudes is satisfied. In the target-space, only one R-R field A_{p+1} is considered other than the usual NS-NS

fields. [3] And we expect that N D_{p+1}-branes are stacked on the boundary to make the U(N) gauge theory there.

Then we start from the following target-space action,

$$
\begin{aligned}
S_D &= \frac{1}{2\kappa^2}\int dx^D\sqrt{|g|}\Big\{e^{-2\Phi}\left(R-4(\nabla\Phi)^2+(\nabla T)^2+V(T)+c\right) \\
&\qquad + \frac{1}{2(p+2)!}f(T)F_{p+2}^2\Big\},
\end{aligned}
\tag{14}
$$

where $c=-(10-D)/2\alpha'$, and $F_{p+2}=dA_{p+1}$ is the field strength of A_{p+1}. Hereafter we take $\alpha'=1$. The total dimension D includes the Liouville direction, which was denoted by ρ. The tachyon potential is represented by $V(T)$, and $f(T)$ denotes the couplings between the tachyon and the R-R field investigated in [16]. When we solve the equations near $T=0$, we use for $V_c(T)=c+V(T)$ its well-known form,

$$V_c=\frac{D-10}{2}-T^2+O(T^4)\,. \tag{15}$$

In the case of super-symmetric world-sheet action, the potential would be the even function of T [16].

Although $f(T)$ is expected to play an important role in the type 0 model [16], we neglect here its T-dependence and set $f(T)=1$ for the simplicity. As for the stability of the tachyon in the dimension $D>2$, it is recovered here by $V(T)$ and D-branes as seen below. Then the equations of motion are written as

$$R_{\mu\nu}-2\nabla_\mu\nabla_\nu\Phi = -\nabla_\mu T\nabla_\nu T+e^{2\Phi}T^A_{\mu\nu} \tag{16}$$

$$-4\nabla_\mu\Phi\nabla^\mu\Phi+2\nabla^2\Phi = \frac{D-2d-2}{4(p+2)!}e^{2\Phi}F_{p+2}^2+V_c(T) \tag{17}$$

$$\nabla^2T-2\nabla_\mu\Phi\nabla^\mu T = \frac{1}{2}V_c'(T) \tag{18}$$

$$\partial_\mu(\sqrt{|g|}F^{\mu\nu_1\cdots\nu_{p+1}})=0\,, \tag{19}$$

where

$$T^A_{\mu\nu}=-\frac{1}{2(p+1)!}\left(F_{\mu\nu_1\cdots\nu_{p+1}}F_\nu^{\ \ \nu_1\cdots\nu_{p+1}}-\frac{g_{\mu\nu}}{2(p+2)}F_{\nu_1\cdots\nu_{p+2}}F^{\nu_1\cdots\nu_{p+2}}\right). \tag{20}$$

We solve the above equations according to the following ansatz;

$$ds^2=e^{2A(r)}\eta_{\mu\nu}dx^\mu dx^\nu+e^{2B(r)}dr^2+e^{2C(r)}\hat{G}_{ab}dx^a dx^b \tag{21}$$

$$\Phi\equiv\Phi(r),\qquad T\equiv T(r)\quad\text{and}\quad A_{01\cdots p}=-e^{q(r)}\ , \tag{22}$$

[3] Although two kinds of the fundamental D-brane can be considered [16], we here consider only the electric type of R-R charge.

where (x^μ, x^a), $\mu = 0 \sim p(= d-1)$ and $a = 1 \sim n$, denote the space-time coordinates. And r is related to the Liouville direction as given below. The equation (19) is solved as

$$\partial_r e^{c(r)} = N e^{dA+B+\bar{d}C} \tag{23}$$

where $\bar{d} = D - d - 1 = D - p - 2$ and N denotes the number of the p-brane. The remaining equations (16) and (17) are solved in the form

$$A(\rho) = \gamma\rho + a(\rho), \qquad B(\rho) = -\rho + b(\rho), \tag{24}$$

$$C(\rho) = C_0 + c(\rho), \quad \Phi(\rho) = \Phi_0 + \phi(\rho), \quad T(\rho) = T_0 + t(\rho), \tag{25}$$

where we used the following notation for the Liouville mode,

$$\rho = \ln r.$$

This relation is given such that the flat metric of $d+1$ sector could be found for $\gamma = 0$, $a = 0$ and $b = 0$ when their coordinates are taken by (ρ, x^μ) as in considering the sigma model.

Then the equations to be solved are given as

$$\ddot{a} + \dot{a}(d\dot{a} - \dot{b} + \bar{d}\dot{c} - 2\dot{\phi}) + \gamma(2d\dot{a} - \dot{b} + \bar{d}\dot{c} - 2\dot{\phi}) = -d\gamma^2 + \frac{\lambda_0^2}{4}e^{2l}, \tag{26}$$

$$d[\ddot{a} + \gamma(2\dot{a} - \dot{b}) + \dot{a}(\dot{a} - \dot{b})] - 2(\ddot{\phi} - \dot{b}\dot{\phi}) + \dot{t}^2 + \bar{d}[\ddot{c} + \dot{c}(\dot{c} - \dot{b})] = -d\gamma^2 + \frac{\lambda_0^2}{4}e^{2l}, \tag{27}$$

$$\ddot{c} + \dot{c}(d\dot{a} - \dot{b} + \bar{d}\dot{c} - 2\dot{\phi}) + d\gamma\dot{c} = (\bar{d} - 1)e^{-2C_0 + 2(b-c)} - \frac{\lambda_0^2}{4}e^{2l}, \tag{28}$$

$$\ddot{\phi} + \dot{\phi}(d\dot{a} - \dot{b} + \bar{d}\dot{c} - 2\dot{\phi}) + d\gamma\dot{\phi} = -\frac{D - 2d - 2}{8}\lambda_0^2 e^{2l} + \frac{1}{2}V_c(T)e^{2b}\,, \tag{29}$$

$$\ddot{t} + \dot{t}(d\dot{a} - \dot{b} + \bar{d}\dot{c} - 2\dot{\phi}) + d\gamma\dot{t} = \frac{1}{2}V_c'(T)e^{2b}\,, \tag{30}$$

where

$$l = b + \phi - \bar{d}(C_0 + c), \quad \lambda_0 = Ne^{\Phi_0}.$$

The dot denotes the derivative with respect to ρ, and λ_0 represents the t'Hooft coupling constant.

4 Solution in linear dilaton vacuum

Here we firstly examine the renormalization group equations of the sigma model only before studying the coupled system of the Yang-Mills theory and the sigma model. The equations are obtained from the above equations by setting as $N = 0$, which means

no D-branes in the background. In this case, the simplest configuration is the linear dilaton vacuum, which is set as

$$\gamma = 0\,, \quad \phi(\rho) = -\frac{1}{2}Q\rho + \varphi(\rho). \tag{31}$$

Now the d-dimensional part is not necessary since the D-branes are absent, then we consider the case $d = 0$ for the simplicity. Further, we take $t(\rho) = 0$, which represents the fluctuation of T around $T = T_0$, and T_0 is specified by

$$V'(T_0) = 0. \tag{32}$$

Then the functions to be solved are b, c and φ. Their equations are obtained from (27) $\sim$ (29) as follows:

$$\bar{d}[\ddot{c} + \dot{c}(\dot{c} - \dot{b})] - Q\dot{b} = 2(\ddot{\varphi} - \dot{b}\dot{\varphi}), \tag{33}$$

$$\ddot{c} + \dot{c}(-\dot{b} + \bar{d}\dot{c} - 2\dot{\varphi}) + Q\dot{c} = (\bar{d} - 1)e^{-2C_0+2(b-c)}, \tag{34}$$

$$\ddot{\varphi} + \dot{\varphi}(-\dot{b} + \bar{d}\dot{c} - 2\dot{\varphi}) + Q(2\dot{\varphi} - \frac{1}{2}\bar{d}\dot{c}) = \frac{1}{2}[Q^2 + V_c(T_0)e^{2b}]\,. \tag{35}$$

The fixed point of the sigma model is easily seen by setting $\dot{c} = 0$ in Eq.(34). Then we find the fixed point at,

$$g_N = e^{-2C} = 0. \tag{36}$$

Further, we should take $\dot{\phi} = 0$ since the linear dilaton vacuum is considered as the fixed point. Then the background charge is obtained from Eq.(35) as $Q = e^{b^*}\sqrt{-V_c(T_0)}$, $V_c(T_0) < 0$ is required. And b^* is a constant as seen from Eq.(33).

We can check the stability for the tachyon fluctuation at this fixed point, where the linearized equation of (30) is written as,

$$\ddot{t} + Q\dot{t} = \frac{1}{2}V_c''(T_0)t\,, \tag{37}$$

where $V_c'(T_0) = 0$ is used. Then we obtain the following condition of stability,

$$V_c''(T_0) \geq \frac{D - 10}{4}\,. \tag{38}$$

When we consider the case $T_0 = 0$, which leads to $V_c''(T_0 = 0) = -2$, the above condition is reduced to $D \leq 2$, which is the well-known result. If we consider to extend this fixed point up to $D = 10$, then we must require $T_0 \neq 0$ and $V_c''(T_0) \geq 0$.

The possible position of this vacuum is shown in Fig.1, which shows the schematic tachyon-potential since the correct form of $V(T)$ is not known. The point B represents the minimum of $V(T)$, and such a point will be found if the second order term of T^4 is positive. But this is open here, and we assume that there is such a point.

Now, we study the running behaviour of g_N near this fixed point. The running solutions are obtained by expanding φ, b and c around this fixed point by an appropriate small parameter. Here two expansion-forms are considered. First, we expand the

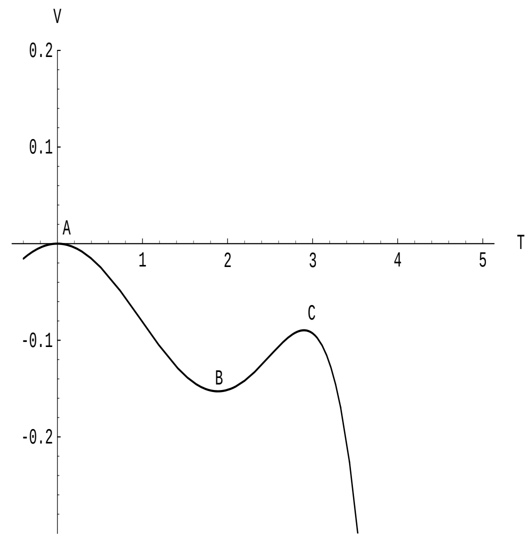

Figure 1: A schematic graph of the tachyon potential $V(T)$, where the scale is arbitrary. The origin **A** represents the perturbative vacuum. The minimum point **B** corresponds to the linear dilaton vacuum, and **C** is the point where the asymptotic free Yang-Mills theory with $\gamma \neq 0$ is realized.

running parameters by g_N as in the usual perturbation since we are considering near $g_N = 0$. Then φ and b are expanded as

$$\varphi = \varphi_1 g_N + \varphi_2 g_N^2 + \cdots, \quad b = b_1 g_N + b_2 g_N^2 + \cdots. \tag{39}$$

And the β-function of g_N is defined as

$$\beta_g = \dot{g_N} = \beta_0 g_N^2 + \beta_1 g_N^3 + \cdots, \tag{40}$$

since the loop-corrections for the coupling constant begins from $O(g_N^2)$. Then the equations (33) $\sim$ (35) are rewritten in terms of β_g, for example (34) can be written as

$$\beta[g_N\beta' + Qg_N - \beta(\frac{\bar{d}}{2} + 1 + 2g_N\varphi' + g_N b')] = -2g_N^3(\bar{d}-1)e^{2b}, \tag{41}$$

where prime denotes the derivative with respect to g_N. From the lower order equations, we obtain

$$\beta_0 = -2\frac{\bar{d}-1}{Q}, \quad \beta_1 = -4\frac{(\bar{d}-1)^2}{Q^3}, \tag{42}$$

$$b_1 = \frac{\bar{d}(\bar{d}-1)}{2Q}, \quad \varphi_1 = -\frac{5\bar{d}(\bar{d}-1)}{16Q^2} + \frac{\bar{d}}{8}. \tag{43}$$

The "dressed" β-function should be defined as $\beta_g = dg_N/\alpha d\rho$, here $\alpha = -1/2(Q - \sqrt{Q^2 + 2V''(T_0)})$, according to the section two. Then we obtain

$$\beta_g = \frac{2}{\alpha Q}(-(\bar{d}-1)g_N^2 - 2\frac{(\bar{d}-1)^2}{Q^2}g_N^3 + \cdots). \tag{44}$$

This is the expected β-function of g_N [11] since it shows the asymptotic freedom and the dressed factor given in (10). The coefficient is exact at least for the one loop result. But the coefficient of the two-loop correction depends of Q even if the dressed factor was moved out. This implies that the relation given in (10) is not correct at two-loop order and the dressed factors are different order by order. The second remark is that the β-function given above is considered as the one of super-symmetric O($\bar{d}$)-nonlinear sigma model. And it is known that the β-function is one-loop finite for O(3) super-symmetric sigma model [24]. But this can not be seen from (44) when we respect to the relation (10), so we should consider that the second order term would be largely modified by the quantum gravity. However the one-loop part is unchanged and Eq.(10) is satisfied. Noticing this point, we continue the analysis further.

Next, we solve the same equations, (33) $\sim$ (35), in terms of another expansions. The asymptotic free coupling constant generally has the behaviour like $1/\rho^a$ with positive constant a near the fixed point, where $\rho \to 0$. Then we expect $c = c_0 \ln \rho + \cdots$, where $\cdots$ tends to zero for $\rho \to \infty$ and $c_0 > 0$. Then by assuming the form,

$$C = C_0 + c_0 \ln \rho + \cdots, \quad b = b_0 \ln \rho + \cdots, \quad \varphi = \varphi_0 \ln \rho + \cdots, \tag{45}$$

we obtain the following solution,

$$c_0 = \frac{1}{2}, \quad b_0 = 0, \quad \varphi_0 = \frac{\bar{d}}{8}, \quad \text{and} \quad \mathrm{e}^{2\mathrm{C}_0} = \frac{2(\bar{\mathrm{d}} - 1)}{\mathrm{Q}}. \tag{46}$$

This result gives the same β_g up to the one loop order, and we can see $g_N \sim 1/\rho$ at large ρ. But the $\varphi(\rho)$, which begins from $\ln \rho$ here, has different behaviour from the one obtained by the g_N-expansions, where $\varphi(\rho)$ begins from $1/\rho$. Both expansions are the perturbative expansions around the same fixed point, so the differences in the ρ dependence of the parameters of the theory could be considered as the difference of the regularization scheme, but the one-loop coefficient of the β-function is scheme independent. We notice here that $\varphi(\rho)$ does not now represent the coupling constant of the gauge theory, which is absent in this case.

The above equations, (33) $\sim$ (35), might be used even at large coupling region. And we would expect an infrared fixed point at some point $g_N = g_N^* \neq 0$, which smoothly continued to the ultra-violet fixed point at $g_N = 0$ given above. If such a fixed point existed, it might be found by using the second type expansions,

$$\beta = \beta_0 (g_N - g_N^*)^{\beta_1} + \cdots, \quad \beta_1 > 0 \tag{47}$$

$$b = b_0 \ln(g_N - g_N^*) + \cdots, \quad \varphi = \varphi_0 \ln(g_N - g_N^*) + \cdots. \tag{48}$$

But we could not find such a solution. Then we could say that the sigma model has only the ultra-violet fixed point at $g_N = 0$.

In the following sections, we consider the case of $\lambda_0 \neq 0$, where the D-branes are in the background and $\phi(\rho)$ represents the gauge coupling constant. And we see how the above critical behaviors of the sigma model will be changed.

5 Solution in background with D-branes

Here we consider the equations, (26) $\sim$ (30), for the case of $N \neq 0$ and $\gamma \neq 0$. Then the terms of the D-branes are included in the equations. The fixed point of g_N and $\lambda = Ne^{\Phi}$ is simply seen by taking them as constants, g_N^* and λ^*. Then we obtain the following relations from Eqs.(28) and (29) by neglecting their left hand sides,

$$(\bar{d}-1)g_N^* = \frac{\lambda^2}{4} g_N^{*\bar{d}}, \tag{49}$$

$$(D-2d-2)\frac{\lambda^2}{4} g_N^{*\bar{d}} = V_c(T^*)\,. \tag{50}$$

From these, we obtain

$$\lambda^{*2} g_N^{*\tilde{d}} = 4\tilde{d}, \tag{51}$$

where $\tilde{d} = \bar{d} - 1 = D - d - 2$. We consider the case of $\tilde{d} > 0$. The relation (51) implies a kind of duality of λ^* and g_N^*. The asymptotic freedom of gauge coupling constant could be seen in the large coupling region of g_N and vice versa. This relation would be satisfied in quite general $d+1$-dimensional manifold since Eq.(51) is independent on $a(\rho), b(\rho)$ and γ which are responsible for the geometry of the $d+1$-dimensional manifold. It is possible to determine the geometry of the background manifold for the fixed point satisfied (51) by solving Eqs.(26) and (27) with respect to $a(\rho)$ and $b(\rho)$. By assuming $t = 0$, we can solve them as

$$\dot{a} = \frac{1}{4}\lambda^{*2} g_N^{*\bar{d}} e^b - \gamma, \tag{52}$$

where $b(\rho)$ is arbitrary. When b is remained as unknown function of ρ, the analysis below becomes very obscure. Then we make an anzatz that $a (= a^*)$ and $b (= b^*)$ are constant at the fixed point also. By taking further as $\dot{t} = 0$, we obtain from the remaining equations

$$d\gamma^2 = \frac{\lambda^2}{4} g_N^{*\bar{d}} e^{2b^*}, \tag{53}$$

$$V_c'(T^*) = 0\,. \tag{54}$$

These results are obtained by neglecting the ρ-dependence of all fluctuations, $\chi_i = \{ a(\rho), b(\rho), \phi(\rho), t(\rho) \}$, from the biggining. And we arrive at the fixed point configuration, $AdS_{d+1} \otimes S^n$.

We firstly solve these near $T = 0$, then $T^* = T_0 = 0$ is obtained from Eqs.(54) and (15). This corresponds to the point A in Fig.1 of the tachyon potential. Other parameters are determined by solving the remaining equations. Possible points examined here are shown in $D - d$ plane of Fig.2, but all these points are not stable. The stability condition is obtained by solving the linearized equation of Eq.(30) with respect to t such that it has a real solution in the form of $t = e^{\alpha\rho}$, i.e. α being real. The equation is given as

$$\ddot{t} + d\gamma\dot{t} = -te^{2b^*}, \tag{55}$$

then the requirement is expressed by,

$$(d\gamma/e^{b^*})^2 \geq 4, \tag{56}$$

since α is obtained as

$$\alpha = \frac{1}{2}(-d\gamma \pm \sqrt{(d\gamma)^2 - 4e^{2b^*}}). \tag{57}$$

After all, we find three stable solutions which satisfy the above condition (56), and they are shown by disks in the Fig.2. The parameters for these solutions are summarized in the Table 1.

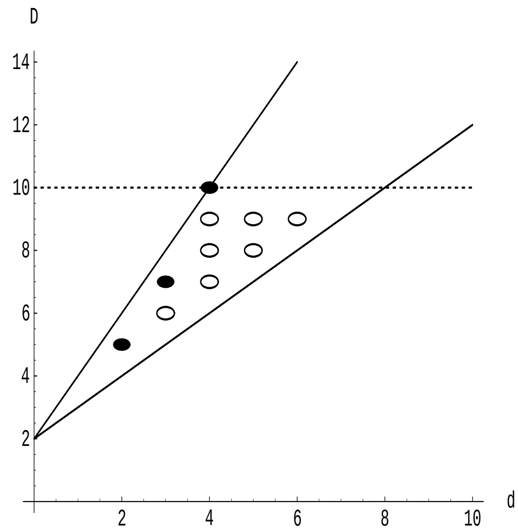

Figure 2: The circles are unstable against tachyon perturbation and the disks are the stable fixed points. The lines represent (a) $D = 2(d+1)$ and (b) $D = d+2$.

	(D,d)	λ^*	g_N^*	γ/e^{b^*}	n
sol.1	(10,4)	$4(e^{b^*}/\gamma)^4$	$(\gamma/e^{b^*})^2$	any constant $\geq 1/2$	5
sol.2	(7,3)	$8\sqrt{2}/3$	$\frac{3}{4}$	$\frac{1}{\sqrt{2}}$	3
sol.3	(5,2)	$2\sqrt{2/5}$	$\frac{5}{2}$	$\frac{\sqrt{5}}{2}$	2

Table 1. The stable fixed points.

We give several comments on these solutions: The (sol.1) is special since Eq.(50) does not give any constraint in this case, and all the values of parameters are not determined. But they are bounded from the stability as, $g_N^* \geq 1/4$ and $\lambda^* \leq 64$. And we can see the relation of these two critical couplings,

$$g_N^{*2}\lambda^* = 4, \tag{58}$$

which is independent of γ/b^*. Although g_N^* approaches to zero with decreasing γ, it is however bounded from below due to the stability and it can not arrive at zero. Then it would be impossible to see the asymptotic free region of the sigma model in the background with D-branes and $T_0 = 0$. This can be interpreted as that the fixed point observed at $g_N = 0$ in the linear dilaton vacuum was pushed to the finite value by D-branes or Yang-Mills theory. If γ decreased across the lower bound, the the tachyon would condense to stabilize the system and a new fixed point will be found in the different background where $T_0 \neq 0$.

On the other hand, it will be possible to consider the limit of $\lambda^* = 0$ in this vacuum, and this is the asymptotic free limit of the gauge theory. In this case, the equations for the running couplings could be solved by expanding them in the series of λ as done in the previous section for g_N. In performing this expansion, we should notice $g_N^* \to \infty$ and $e^{b^*} \to 0$ for $\lambda^* \to 0$. Then we parametrize them as

$$b(\rho) = b_0 \ln \rho + b_1/\rho + \cdots, \quad c(\rho) = c_0 \ln \rho + c_1/\rho + \cdots, \tag{59}$$

and introduce $\beta_\lambda = d\lambda/d\rho$ which is expected to be expanded as $\beta_\lambda = \beta_0 \lambda^3 + \cdots$. However we can not find any solution in this form. The reason why this expansion has failed would be reduced to the fact that the right hand side of Eq.(29) vanishes. But the terms of the right hand side are necessary to obtain a desired renormalization group flow of the Yang-Mills theory. This point can be understood from the solution obtained in the previous section, where the first term of the right hand side of Eq.(28) was necessary to obtain the expected asymptotic free β-function of the sigma model. In fact also for λ, the asymptotic free solutions are seen by using the above expansion in the next section where the second term of the right hand side of Eq.(29) is retained by considering the vacuum with $T_0 \neq 0$. Then the running behaviours of the solution (sol.1) can be seen in a different expansion-form which is suitable to the case of the finite coupling constant.

For solutions (sol.2,3), the parameters are all determined definitely. The value of b^* depends on γ, but λ^* and g_N^* are fixed independently on γ. Then the coupling constants at the fixed points are fixed definitely, this is due to availability of the Eq.(50). Although the field theory on the branes are different from the 4d Yang-Mills theory, we can expect the similar features of the flow equations to the case of (sol.1). As shown below, these are the ultra-violet fixed points for both sigma model and the field theory on the branes.

Running behaviours : First we consider the (sol.1), where we can see the relation of the sigma model and the 4d Yang-Mills theory. Their coupling constants at fixed points are related by Eq.(58). Near the fixed point, the equations can be approximated by the linearized one;

$$\ddot{a} + \gamma(2d\dot{a} - \dot{b} + \bar{d}\dot{c} - 2\dot{\phi}) = \frac{\lambda_0^2}{2}(\phi - \bar{d}c + b), \tag{60}$$

$$d\ddot{a} + d\gamma(2\dot{a} - \dot{b}) + \bar{d}\ddot{c} - 2\ddot{\phi} = \frac{\lambda_0^2}{2}(\phi - \bar{d}c + b), \tag{61}$$

$$\ddot{c} + d\gamma\dot{c} = -\frac{\lambda_0^2}{2}[\phi - (\bar{d}-1)c], \tag{62}$$

$$\ddot{\phi} + d\gamma\dot{\phi} = 0\,, \tag{63}$$

$$\ddot{t} + d\gamma\dot{t} = \frac{1}{2}V_c''(T_0)t\,. \tag{64}$$

First, we can solve Eqs.(62) and (63) with respect to c and ϕ. They represent how two dynamical system, 2d sigma model and 4d Yang-Mills theory, are interacting. Once some non-trivial c is given, the running behaviour of ϕ is given from Eq.(62). In other words, the β-function of the Yang-Mills theory is induced by the 2d sigma model and vice versa. By using ϕ and c determined by Eqs.(62) and (63), a and b are obtained from Eqs.(60) and (61). The equation of t, (64), is separated alone, then it does not give any essential effect to the running couplings of two theories.

Eq.(63) is solved as $\phi = \phi_1 e^{-d_0\rho}$, where $d_0 = d\gamma$ and ϕ_1 is some constant. Then solutions for χ_i can be solved systematically order by order in terms of the following expansions,

$$\chi^i(r) = \sum_{n=1}^{\infty} \chi^i_n (e^{-d_0\rho})^n\,. \tag{65}$$

For example, $a(r) = \sum_n a_n(e^{-d_0\rho})^n$. Up to the second order, the coefficients $\{\ c_1, c_2\ \}$ and $\{\ a_1, a_2\ \}$ are taken arbitrarily, and the one of ϕ, b and t are represented by them:

$$\phi_1 = 4c_1, \quad \phi_2 = 16c_1(c_1 - a_1); \quad t(\rho) = O(e^{-3\lambda_0\rho}), \tag{66}$$

$$b_1 = 5c_1 - 4a_1, \quad b_2 = -8a_2 - \frac{41}{3}c_2 + \frac{8}{3}(29c_1^2 - 28c_1a_1 - 3a_1^2). \tag{67}$$

From these perturbative results, the β-function, which is defined as $\beta(\lambda) \equiv d\lambda/d\rho$ for $\lambda(= Ne^{\Phi} = Ng_{YM}^2)$, satisfies the following relation at the critical point:

$$\beta'(\lambda^*) = -d_0 = d\gamma. \tag{68}$$

For $g_N = e^{-2C}$, we obtain $\beta'(g_N^*) = -2d_0/|\alpha|$ by re-scaling the scale parameter, where α is given in (57). This means that the fixed point given above is the ultra-violet one for both 2d sigma model and the 4d Yang-Mills theory.

In the case of $D < 10$, there are two stable solutions at $(D, d) = (7, 3)$ and $(5, 2)$ as shown in the Table 1. In both cases, the d-dimensional field theories are not the 4d Yang-Mills theory and Φ would not correspond to the running coupling-constant of the Yang-Mills theory. But we can see the similar relations between the sigma model and the theory in the branes as in the case of $D = 10$. They affect each other and the fixed points of both theories are at finite values as given in the Table 1. As for the linearized equations of χ^i are the same with the one of $D = 10$ except for Eq.(63), which is written here by

$$\ddot{\phi} + d\gamma\dot{\phi} = -\frac{D-10}{2}(\phi - \bar{d}c)\,. \tag{69}$$

The equations are solved as in the case of $D = 10$. First, this equation and Eq.(62) are solved in the form of $\phi = \phi_1 e^{\delta\rho}$ and $c = c_1 e^{\delta\rho}$, then we obtain

$$\delta = -1 \pm \sqrt{E_\pm}, \qquad E_\pm = \frac{1}{2}(D - 2 \pm \sqrt{D^2 - 12D + 84}). \tag{70}$$

The value of δ are evaluated for the two cases as

$$\delta = \begin{cases} -4.30,\ 2.07,\ -1.12 \pm 0.39i & \text{for (D,d)=(5,2)} \\ -4.09,\ -1.81,\ -0.308,\ 1.97 & \text{for (D,d)=(7,3).} \end{cases} \tag{71}$$

It should be real and negative since we are considering at the ultra-violet fixed point. In this sense, we can choose $\delta = -4.30$ for $(D, d) = (5, 2)$, but there are three candidates in the case of $(D, d) = (7, 3)$ and there is no principle to choose one of them. Including this problem, these two solutions are remained to be explained in more detail based on some meaningful dynamical system.

6 Asymptotic free solution

The fixed points considered above correspond to a finite value of the coupling constants for both gauge theory and the sigma model. However we can expect a fixed point where asymptotic freedom is seen for one of them due to the relation (51). Such a solution would be found by solving the equations in terms of the following expansion for χ^i [25, 26]. For a, we take the form,

$$a(\rho) = \bar{a}_0 \ln \rho + \bar{a}_1 \frac{1}{\rho}(\ln \rho + \bar{a}_{10}) + \cdots, \tag{72}$$

For b and c, they are written by replacing $(a,\ \bar{a}_0\ \bar{a}_{10})$ by $(b,\ \bar{b}_0,\ \bar{b}_{10})$ and $(c,\ \bar{c}_0,\ \bar{c}_{10})$ respectively in the above equation (72). And we set

$$\phi(\rho) = \bar{\phi}_0 \ln \rho + \bar{\phi}_1 \frac{1}{\rho} \ln \rho + \bar{\phi}_2 \frac{1}{\rho^2} \ln^2 \rho + \cdots, \tag{73}$$

$$t(\rho) = \bar{t}_1 \frac{1}{\rho} + \bar{t}_2 \frac{1}{\rho^2}(\ln \rho + \bar{t}_{20}) + \cdots. \tag{74}$$

In these functional form, the asymptotic free gauge theory is obtained for $\bar{\phi}_0 < 0$. While the asymptotic free behaviour of the 2d sigma model will be found for $\bar{c}_0 > 0$.

6.1 Solutions for $\gamma \neq 0$

First, we consider the case of $\gamma \neq 0$. The linearized equations with respect to the fluctuations χ^i are not useful when we use the above expansions, and full form of equations are now used. And they are solved by expanding equations in the power

series of $1/\rho$, which is assumed being very small. Firstly the lowest order terms are examined. From Eqs. (26), (27) and (28), we obtain

$$\frac{\lambda_0^2}{4} e^{-2\bar{d}C_0} = d\gamma^2 = (\bar{d}-1)e^{-2C_0}, \tag{75}$$

$$\bar{\phi}_0 = (D-d-2)\bar{c}_0, \quad \bar{c}_0 = \bar{b}_0. \tag{76}$$

The above solution is possible only for $D-d-2>0$ since we are considering the case $\gamma \neq 0$. From (29) if $D \neq 2(d+1)$, then $\bar{b}_0 = 0$ in order to cancel the lowest order term on the right hand side of this equation (29). Then (76) implies the trivial solution, $\bar{\phi}_0 = \bar{c}_0 = \bar{b}_0 = 0$. So, we must set $D = 2(d+1)$. Then the first equation of (76) is written as $\bar{\phi}_0 = d\bar{c}_0$. From this we can see that if Yang-Mills theory is asymptotic free then the coupling constant of the sigma model is asymptotically infinite and vice versa. This is the reflection of the relation given in (51). For $\gamma \neq 0$ case, only the solution of negative $\bar{\phi}_0$ is possible as shown below.

Namely, we obtain $\bar{b}_0 = -1/2$ and $V_c(T_0) = 2d\gamma\bar{\phi}_0$ by considering the next order terms ($O(1/\rho)$) in (29). Then $c_0 = -1/2$, and this implies the asymptotically infinite sigma model coupling constant as pointed out above . The results are summarized as

$$\bar{\phi}_0 = -\frac{D-2}{4}, \quad \bar{c}_0 = \bar{b}_0 = -\frac{1}{2}, \quad D = 2(d+1), \tag{77}$$

$$V_c(T_0) = -\frac{D-2}{2} d\gamma. \tag{78}$$

As for the tachyon potential, we can see more from the lowest and the first order terms of Eq.(30), which leads to

$$V'(T_0) = 0, \quad V''(T_0) = -2d\gamma. \tag{79}$$

The second equation of (79) is derived by assuming $\bar{t}_1 \neq 0$. From these, the fixed points given here will be situated at the maximum point of the tachyon potential; at point $A(T_0 = 0)$ or $C(T_0 \neq 0)$ in the Fig.1.

As for the value of T_0, we can check it from (78) and (15). If $T_0 = 0$, we obtain

$$D = \frac{10 + d\gamma}{1 + d\gamma}. \tag{80}$$

This is not satisfied for $D = 10$ since $\gamma \neq 0$. Then the fixed point for $D = 10$ should be realized at $T_0 \neq 0$ which is identified with the point C, or other local maximum with $T_0 \neq 0$. We give one more remark for the case of $D = 10$. We obtain $\bar{\phi}_0 = -2$ and $d = 4$ from equations of (77), then we can see that the coupling constant of 4d Yang-Mills theory decreases at large ρ as $g_{YM}^2 \propto 1/\rho^2$. But the expected case is $\bar{\phi}_0 = -1$. This point might be resolved by considering an effective coupling constant obtained from the Born-amplitude of $Q\bar{Q}$ scattering [25].

Next, we comment on the solutions for $D < 10$. It is easy to find the solution of $\bar{\phi}_0 = -1$ at $D = 6$. But we find $d = 2$ in this case, so $\bar{\phi}_0 = -1$ can not be connected

to the favorable behaviour of the β-function of the 4d Yang-Mills theory. For $D = 6$ solution, $T_0 = 0$ is possible and all parameters are determined as

$$\lambda_0 = 8\sqrt{2}, \quad \gamma = 1/2, \quad e^{-2C_0} = 1/4. \tag{81}$$

Then this fixed point can be taken at the point A in the Fig.1, and the parameters would be determined definitely. The above solutions are summarized in the Table 2.

	(D,d)	ϕ_0	$\bar{c}_0$	T_0	n
af1	(10,4)	-2	-1/2	$\neq 0$	5
af2	(6,2)	-1	-1/2	could be 0	3

Table 2. Asymptotic free solutions for $\gamma \neq 0$.

Finally we examine the stability of the tachyon fluctuation from its linearized equation of Eq.(30). Near the fixed point, $\rho \to \infty$, the linear term of t is expressed as

$$\ddot{t} + d\gamma\dot{t} = \frac{1}{2\rho} V_c''(T_0)t \,. \tag{82}$$

Since the right hand side is negligible small, then this fixed point is stable against small tachyon fluctuation for the above two solutions.

6.2 Solutions for $\gamma = 0$

Next, we consider the case of $\gamma = 0$. In this case, the bulk configuration will largely deviate from the horizon configuration, $AdS_{d+1} \otimes S^n$, of the original D-branes given in the type IIB string theory. Then we now widely extend the duality relation of bulk gravity and the quantum field theory on the boundary or the branes.

The equations are solved in the form given by (72) $\sim$ (74). Here we notice the following points. In this case also, the term from the D-brane, the term proportional to λ_0, must be suppressed at $\rho \to \infty$ at least to the order of ρ^{-2} to obtain a solution with $\gamma = 0$ as seen from Eqs.(26) and (27). This is because of the fact that the left hand sides of these two equations are the order of ρ^{-2}. The situation is the same for the other three equations. Then the terms of the right hand sides, the first term of (28) (the curvature term of S^n) and the second potential term of (29), will also suppressed as ρ^{-2} or more. Several possibilities are considered here in solving the equations depending on which terms are retained or abandoned in the right hand sides as the one of order of ρ^{-2}. Among those solutions, we show here two interesting cases.

First we consider the case where the terms coming from the D-branes and S^n-curvature are retained and the term from the tachyon-potential is suppressed. Namely, we assume

$$\bar{b}_0 + \bar{\phi}_0 - \bar{d}\bar{c}_0 = -1, \quad \bar{b}_0 - \bar{c}_0 = -1, \quad \bar{b}_0 < -1. \tag{83}$$

Then we obtain

$$\bar{\lambda}_0^2 = 2(D-2)\bar{a}_0^2, \quad e^{-2C_0} = (\frac{D-2}{2(\bar{d}-1)})^2 \bar{a}_0^2, \tag{84}$$

where $\bar{\lambda}_0 = \lambda_0 e^{-2\bar{d}C_0}$ and

$$\bar{\phi}_0 = (d+1-\frac{D}{2})\bar{a}_0, \quad \bar{c}_0 = \frac{\bar{\phi}_0}{\bar{d}-1}, \quad \bar{b}_0 = \bar{c}_0 - 1. \tag{85}$$

Here $\bar{a}_0$ remains undetermined. Since $\bar{b}_0 < -1$, we can see that $\bar{\phi}_0$ and $\bar{c}_0$ should be negative. This solution then could provide the desirable asymptotic free 4d Yang-Mills theory if we set as $\bar{\phi}_0 = -1$ and $d = 4$, which is possible here. In order to proceed the systematic perturbation in solving the equations, the right hand sides of the equations to be solved should be expanded by the power series of $1/\rho$. For the above setting, we obtain the integer value of $\bar{b}_0$ for $D = 7$ as $\bar{b}_0 = -(D-5)/(D-6) = -2$. And other parameters are determined as $\bar{a}_0 = -2/3, \bar{c}_0 = -1$ and

$$\bar{\lambda}_0 = -\frac{2}{3}\sqrt{10}, \quad e^{-C_0} = \frac{5}{6}. \tag{86}$$

From these we obtain the β-function defined as $\beta_\lambda = d\lambda/d\rho$ for the t'Hooft parameter $\lambda = Ne^{\Phi}$ as

$$\beta_\lambda = -\frac{25}{24\sqrt{10}}\lambda^2 + O(\lambda^3). \tag{87}$$

This result is the expected one for the 4d Yang-Mills theory, but the coefficient $\beta_0 = -\frac{25}{24\sqrt{10}}$ might be adjusted by the factor coming from the relation between ρ and the mass scale in the 4d Yang-Mills theory. In this sense, the relative ratios of the coefficients will be quantitatively important.

As for the sigma model, its coupling constant is asymptotically infinite and we have no knowledge to compare with this result. Then we comment only on the relation of the mass scale $(1/a)$ in the 2d sigma model and ρ. When we respect the relation $\rho = 2\ln a/\alpha$, we must determine α from the linealized equation of tachyon fluctuation t as setting $t = e^{\alpha\rho}$. However, we obtain $\alpha = 0$ due to $\gamma = 0$. If we set as $\alpha = 0^-$, the large variation of ρ corresponds to a very small change of $1/a$ and the coupling constant does not run with respect to a.

We show the second interesting solution. It is obtained when the terms proportional to λ_0 and V_c are retained, and the term of S^n is neglected. This situation is realized by the relations,

$$\bar{b}_0 + \bar{\phi}_0 - (D-d-1)\bar{c}_0 = -1, \quad \bar{b}_0 = -1, \quad \bar{b}_0 - \bar{c}_0 < -1. \tag{88}$$

In this case, we obtain

$$V_c(T_0) = -\frac{D}{4}\bar{\lambda}_0^2, \tag{89}$$

$$\bar{\phi}_0 = \frac{\bar{d}}{2\sqrt{D-1}}\bar{\lambda}_0, \quad \bar{c}_0 = \frac{1}{2\sqrt{D-1}}\bar{\lambda}_0 = -\bar{a}_0. \tag{90}$$

In this solution, both $\bar{\phi}_0$ and $\bar{c}_0$ are positive, then 2d sigma model is asymptotic free but λ is asymptotically infinite. This solution was expected in the section five as the limit of $\gamma \to 0$ of the (sol1), where $(D,d) = (10,4)$. Here several values for (D,d) are possible, but we do not consider this solution furthermore since the situation of the mass scale for the sigma model is very different from the case of the linear dilaton vacuum as mentioned above. The value of T_0 should be nonzero since $V_c(T_0)$ should be negative and finite as seen from Eq.(89). Then the fixed points given here would be situated at point B or C in the Fig.1. The above two solutions are summarized in the Table 3.

	(D,d)	$\bar{\phi}_0$	$\bar{c}_0$	T_0	n
af3	(7,4)	-1	-1	could be 0	2
af4		positive	positive	$\neq 0$	

Table 3. Asymptotic free solutions for $\gamma = 0$.

As for the stability of the above two solutions, there is no problem in the sense that the linearized equation of t does not provide imaginary solution. The term proportional to λ_0, which is representing the D-branes, was essential to obtain the above two solutions. This point would be important to obtain a realistic critical behaviour of the Yang-Mills theory, but the solution of $\bar{\phi}_0 = -1$ is restricted to $(D,d) = (7,4)$.

We finally comment on the fact that we can get more various solutions when we take the conditions,

$$\bar{b}_0 + \bar{\phi}_0 - (D-d-1)\bar{c}_0 < -1, \quad \bar{b}_0 < -1, \quad \bar{b}_0 - \bar{c}_0 < -1, \tag{91}$$

by which we can neglect all the terms on the right hand sides of the Eqs.(26) $\sim$ (30) at least in the lowest order. The order of these neglected terms are determined by solving the equations of lower order, where these terms can be neglected, and the consistency of the results with the above assumptions are assured. But we do not consider this case more since it seems to be unnatural to determine the lowest order part of the renormalization group equations without D-branes or S^n curvature.

7 Conclusion

We have examined the critical behaviours of the Yang-Mills theory and the 2d sigma model through the non-critical string theory, which is based on the super-symmetric world-sheet action, in the context of the holography. The Yang-Mills theory lives on

the D-branes at the boundary of the bulk-space and the sigma model can be considered on the world-sheet of the non-critical string which could propagate in the bulk-space. They are coupled each other through the bulk gravity. The classical solutions of the gravity with D-branes provides the renormalization group flows for both theories simultaneously, and the Liouville field plays the role of the common mass scale of the renormalization group equations in both theories.

The fixed points found here can be specified by the vacuum value of the tachyon (T) and the position of its potential. Since the potential is not known except for the region near $T = 0$, a schematic tachyon-potential $V(T)$ has been used for the help of understanding. In the analysis of the sigma model, the D-branes can be neglected and we obtain the expected β-function in the linear dilaton vacuum which is specified at the minimum of $V(T)$ with $T \neq 0$. When D-branes are present, we obtain renormalization group equations for both theories, 2d sigma model and 4d Yang-Mills theory. The fixed points are obtained by neglecting the scale-parameter dependence of the running couplings. We find that the coupling constants of the two theories are not independent and their product is finite at the fixed point. This relation implies that the asymptotic freedom of one theory can be seen in the region where the coupling of the other theory is asymptotically infinite. We could find an explicit solution which shows the asymptotic free behavior of 4d Yang-Mills theory as expected from the perturbation of the field theory. This solution is obtained in the limit where the AdS like geometry disappears. On the other hand, the coupling constant of the sigma model is very large, and it can not be "running" when the mass-scale is replaced by the physical one of the 2d sigma model.

Many problems are remained to be resolved. There are many other kinds of fixed points and renormalization group flows for both the sigma model and Yang-Mills theory. They are classified by three schematically typical positions of the tachyon potential. Before studying physical properties of these fixed point, it would be necessary to find a reliable form of the potential in order to accept these fixed points and the renormalization group flows as the realistic one. Secondly, we have neglected the running of the tachyon field for the simplicity. If we could find a non-trivial running solution of the tachyon, which could connect two positions corresponding to the different-type fixed points of the Fig.1, it might be a solution of a renormalization group flow connecting two fixed points. This is an open problem here. Further, we should resolve the usefulness of our analysis at small 'tHooft coupling region where string-loop corrections are usually expected.

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